1.5 Electric Potential

From vector analysis it is well-known that if the curl of a vector vanishes, then the vector may be expressed as the gradient of a scalar. It is shown below that $\vec{\nabla} \times \vec{E} = 0$. This means that the electrostatic field $\vec{E}$ is conservative and it can be written as the gradient of some scaler ϕ as $\vec{E} = -\vec{\nabla}\phi$, where the negative sign is chosen for convenience. The scalar function ϕ thus introduced is known as *electrostatic potential.*

To show that $\vec{\nabla} \times \vec{E} = 0$ we first consider the electric field at the position $\vec{r}$ due a point charge q located at $\vec{r}'$. This is

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}. \tag{1.5-1}$$

Substituting these results in Eq. (1.5-2) we get

$$\bar{\nabla} \times \vec{E} = 0. \tag{1.5-3}$$

From the principle of superposition we known that the total field due to any charge distribution is given by the vector sum of the fields due to individual charges. So we can say that the result (1.5-3) holds for any charge distribution. Therefore, we can write

$$\vec{E} = -\vec{\nabla}\phi. \tag{1.5-4}$$

Since $\vec{\nabla}(\phi + constant) = \vec{\nabla}\phi$, the potential ϕ defined by (1.5-4) is arbitrary by some additive constant. Addition of a constant to ϕ yields the same electric field. Moreover, it does not affect the potential difference between two points because the constants cancel out. The absolute value of potential is of no importance, only potential differences have physical significance. To get a physical significance of the electrostatic potential let us

physical significance. To get a physical significance of the electrostatic potential let us calculate the work done against the field in moving a unit positive charge from a reference point a to the point b. This is

$$W_{ab} = - \text{ work done by the field}$$

$$= - \int_a^b \vec{E} \cdot d\vec{r} = \int_a^b \vec{\nabla}\phi \cdot d\vec{r} = \int_a^b d\phi = \phi(b) - \phi(a). \qquad (1.5\text{-}5)$$

Obviously the work done depends only on the positions a and b and not on the path connecting them. This indicates that the work done over any close path is zero. This result can be easily obtained from Eq. (1.5-3) by using Stokes' theorem,

$$\oint_C \vec{E} \cdot d\vec{r} = \int_S \left(\vec{\nabla} \times \vec{E}\right) \cdot d\vec{S} = 0, \qquad (1.5\text{-}6)$$

where C is the contour bounding an open surface S.

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This must be so because electrostatic force is conservative. Equation (1.5-5) suggests that the electrostatic potential may be considered as potential energy per unit charge. If the charge distribution that creates the electric field is localised in a finite region of space the electric field vanishes at infinite distances from the charge distribution. Then one usually takes the reference point a at infinity, where the potential is taken to be zero; $\phi(\infty) = 0$. In this case Eq. (1.5-5) gives

$$\phi(b) = -\int_{\infty}^{b} \vec{E} \cdot d\vec{r}. \tag{1.5-7}$$

Now, as the electric field is the force per unit positive charge, *the electrostatic potential at any point* (b) *may be defined as the work done by an external agency in bringing a unit positive charge from infinity to that point.*

If the same reference point is chosen for the electrostatic potential and potential energy then the potential energy of a charge is given by the product of the charge and the electric potential at the location of the charge.

Potential due to a point charge

Electric field at the position $\vec{r}$ due to a charge q located at $\vec{r}'$ is given by the Eq. (1.5-1). Now from vector analysis it can be shown that

$$\vec{\nabla} \frac{1}{|\vec{r} - \vec{r}'|} = -\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}; \quad \text{where } \vec{\nabla} \text{ operates on } \vec{r}.$$

Therefore, Eq. (1.5-1) can be rewritten as,

$$\vec{E} = -\vec{\nabla} \left[\frac{1}{4\pi\epsilon_0} \cdot \frac{q}{|\vec{r} - \vec{r}'|} \right]$$

Comparing it with the relation $\vec{E} = -\vec{\nabla}\phi$ we get the electrostatic potential due to the point charge as

$$\phi = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{|\vec{r} - \vec{r}'|}. \tag{1.5-8}$$
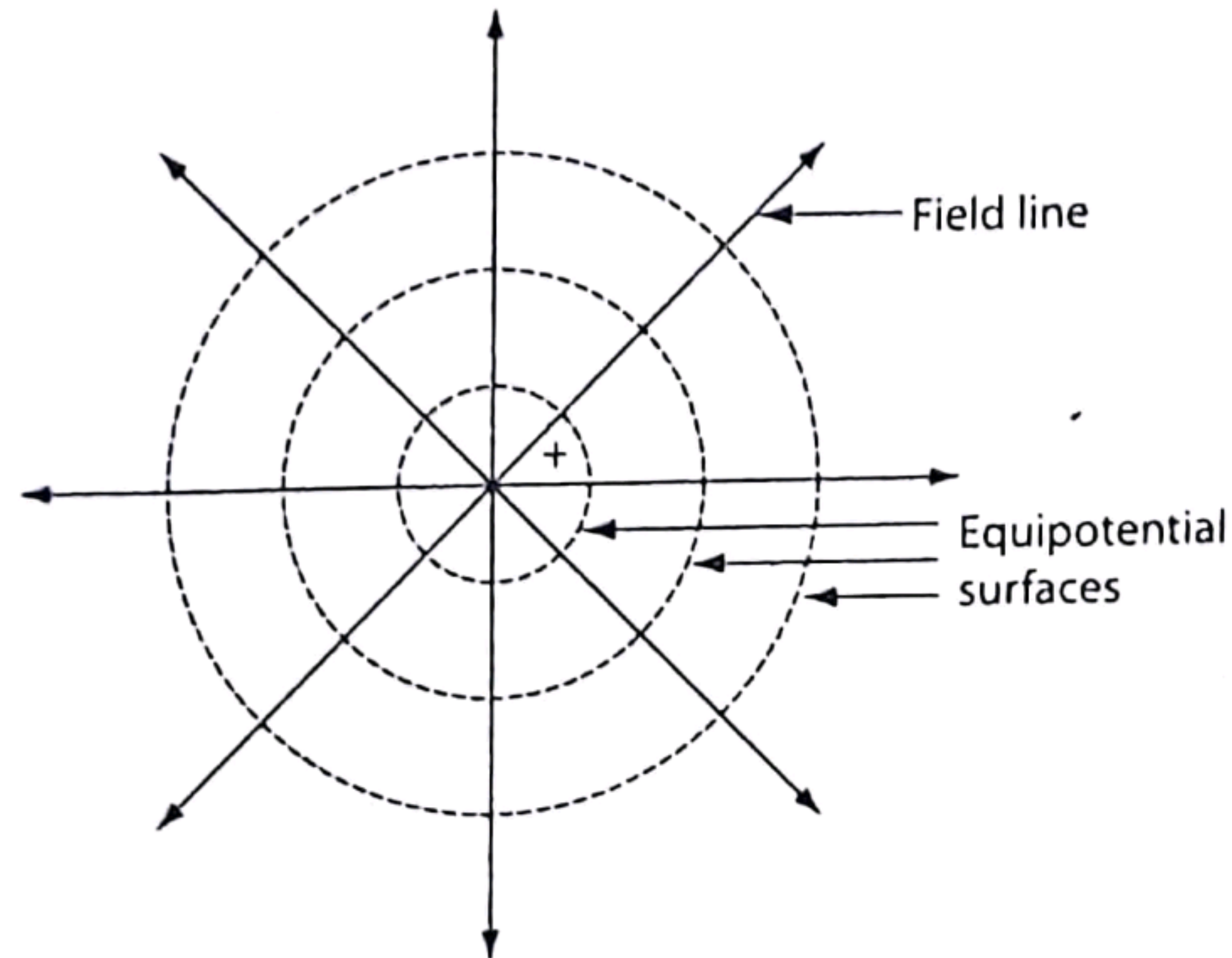
Equipotential surfaces

An equipotential surface is the locus of points having the same potential. In case of a point charge q located at the origin, the potential at a distance r from the origin is given by

$$\phi(\bar{r}) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r}.$$

Obviously a sphere of radius r with the centre at q will be an equipotential surface. In this case, all concentric spheres with the centre at q will be equipotential surfaces

Obviously a sphere of radius r with the centre at q will be an equipotential surface. In this case, all concentric spheres with the centre at q will be equipotential surfaces (Fig 1 5-1). In general the shape of the equipotential surface depends on the charge configuration. But whatever may be its shape the relation $\vec{E} = -\vec{\nabla}\phi$ indicates that the *electric field $\vec{E}$ is always perpendicular to the equipotential surface, $\phi = $ constant, at every point.*

Fig 1.5-1: Field lines and equipotential surfaces for a point charge.

Advantage of the potential concept

Electric field $\vec{E}$ is a vector quantity. Its direct calculation very often becomes tedious and cumbersome. On the other hand, potential ϕ is a scalar quantity. In many cases its calculation is found to be easier. The potential concept reduces a vector problem down to a scalar one. So in practice it is often preferable to determine $\vec{E}$ by first calculating ϕ and then using the relation $\vec{E} = -\vec{\nabla}\phi$, instead of determining $\vec{E}$ directly.
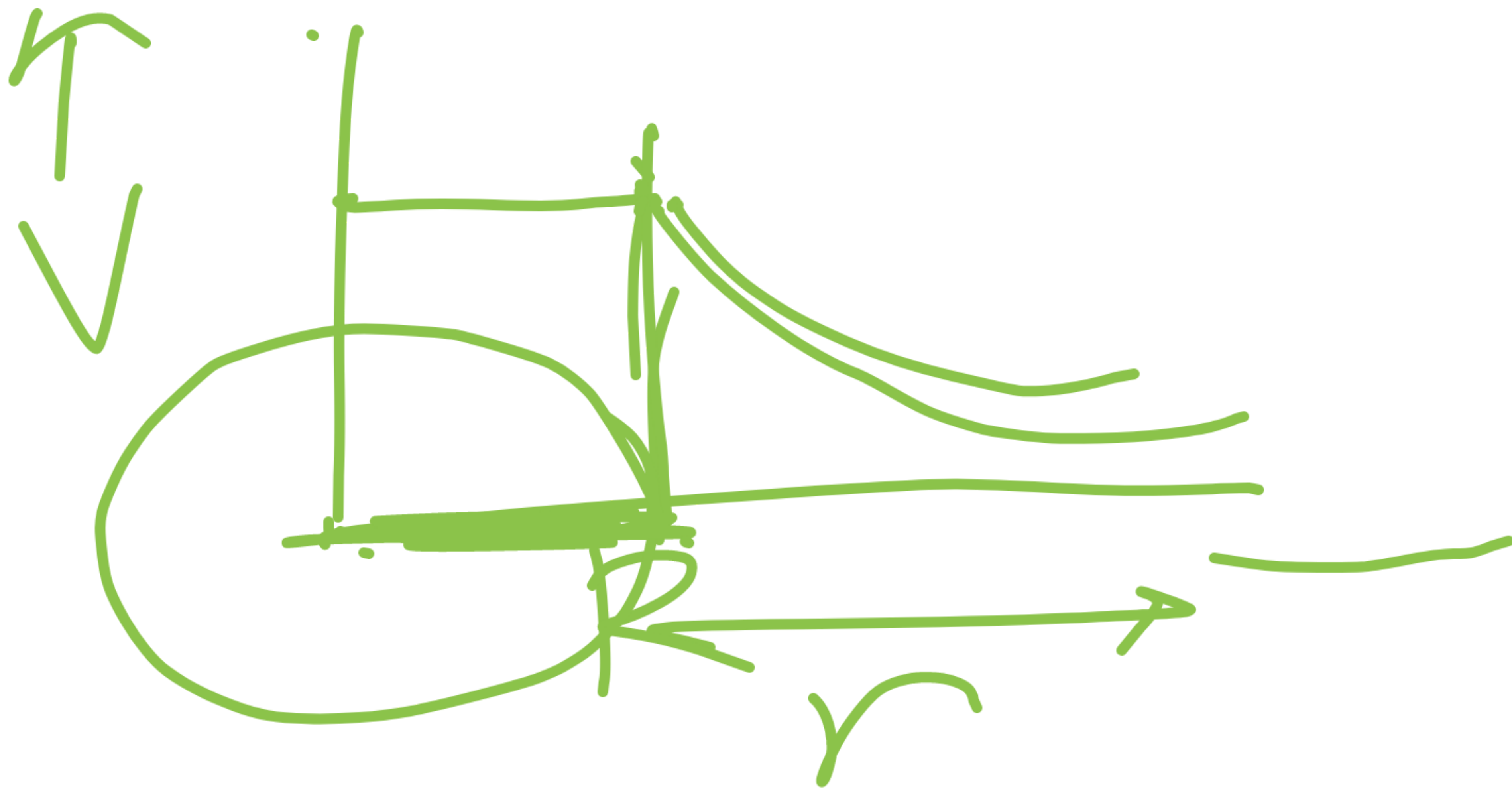
shell $r > R$

$$\phi = \frac{Q}{4\pi r} = \frac{Q}{r}$$
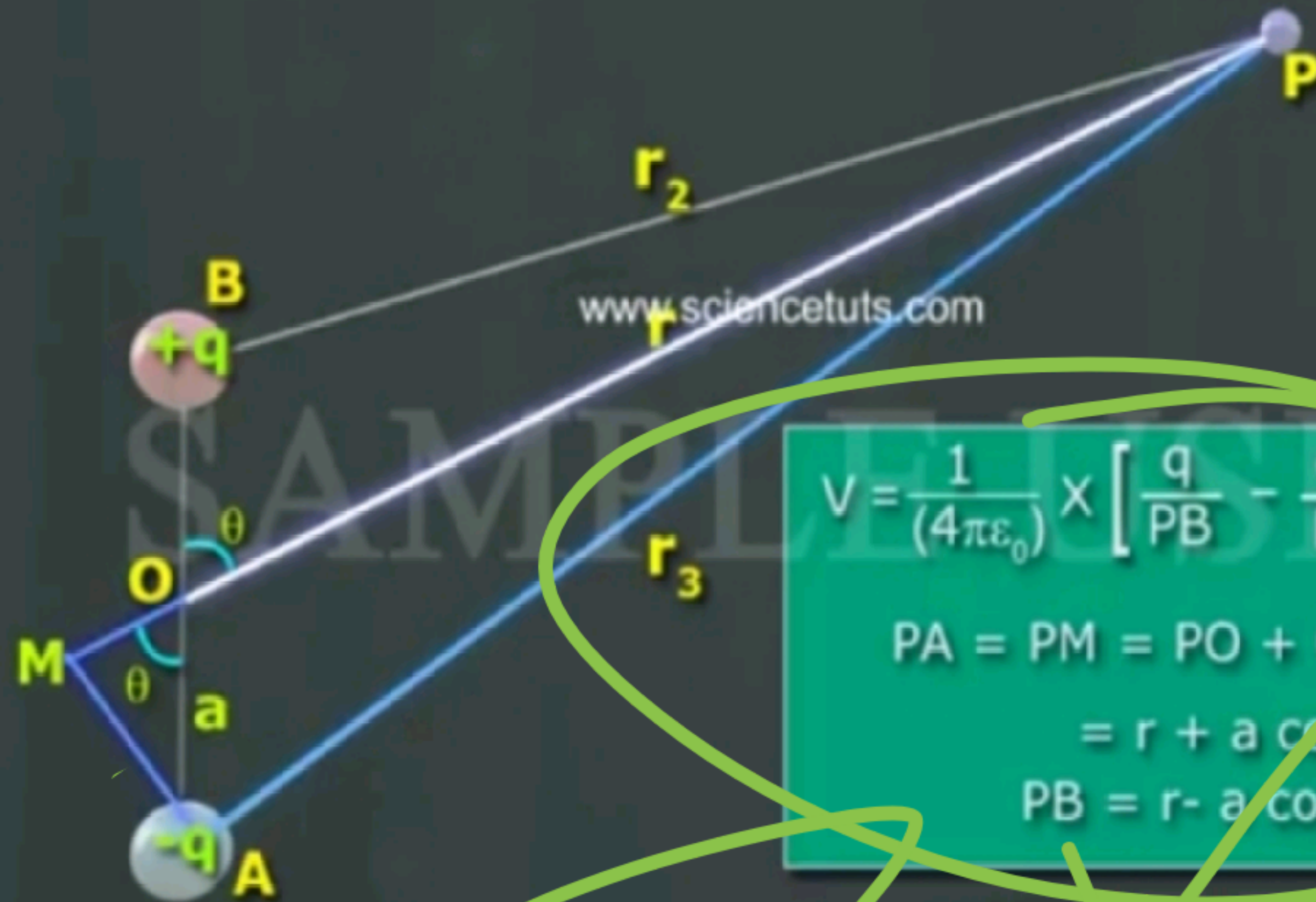
$$\phi = k \frac{Q}{R}$$

$$E = 0$$

$$\phi = const.$$

$$\Delta \phi = 0$$

POTENTIAL DUE TO AN ELECTRIC DIPOLE

r_2

B
+q

r

θ

O

M

θ

a

r_3

-q
A

P

$$V = \frac{1}{(4\pi\varepsilon_0)} \times \left[\frac{q}{PB} - \frac{q}{PA} \right]$$

$$PA = PM = PO + OM$$

$$= r + a\cos\theta$$

$$PB = r - a\cos\theta$$

Substituting in above relation

$$V = \frac{1}{4\pi\varepsilon_0} q \left[\frac{1}{r - a\cos\theta} - \frac{1}{r + a\cos\theta} \right]$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q^2 a\cos\theta}{r^2 - a^2\cos^2\theta}$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q^2 a\cos\theta}{r^2\left[1 - \frac{a^2\cos^2\theta}{r^2}\right]}$$

When, $r \gg a$

$$V = \frac{1}{4\pi\varepsilon_0} \times \frac{P\cos\theta}{r^2}$$

$$V = V_1 + V_2$$

$$V_1 = q \quad V_2 = q$$

$$V_1 = k \cdot \frac{-q}{PA} \quad V_2 = k \frac{+q}{PB}$$

H.I

Solid
Sphere
potential?

e.